Introduction

A general second-order differential equation with variable coefficients is of the form

\[ a_2(t) y'' + a_1(t) y' + a_0(t) y = f(t) \]  

(1)

or, written in the standard form

\[ y'' + p(t) y' + q(t) y = g(t) \]  

(2)

Existence and Uniqueness Theorem:

If the function \( p(t) \), \( q(t) \) and \( g(t) \) are continuous on an interval \((a, b)\) that contains the point \( t_0 \), then for any choice of the initial values \( y_0 \) and \( y_1 \), there exists a unique solution on the interval \((a, b)\) of the IVP:

\[ y'' + p(t) y' + q(t) y = g(t) ; \quad y(t_0) = y_0 ; \quad y'(t_0) = y_1. \]  

(3)

Homogeneous Equations

A condition for the linear independence of solutions:

Two solutions \( y_1(t) \) and \( y_2(t) \) (defined on the interval \( I \)) of the homogeneous differential equation

\[ y'' + p(t) y' + q(t) y = 0 \]  

(4)

are linearly independent if and only if the Wronskian

\[ W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = y_1(t)y'_2(t) - y_2(t)y'_1(t) \]

is non-zero on the whole interval \( I \).

\[ \rightarrow \] If two linearly independent solutions \( y_1(t) \) and \( y_2(t) \) of the homogeneous equation (4) are known, the general solution of the homogeneous equation (4) is \( y(t) = c_1 y_1(t) + c_2 y_2(t) \).

\[ \rightarrow \] In general, finding two linearly independent solutions is not an easy task. However, if one particular solution \( y_1(t) \) of the homogeneous equation (4) is known, a second linearly independent solution \( y_2(t) \) may be found using the Reduction of Order method.

Reduction of Order:

Let \( y_1(t) \) a non-trivial solution of the homogeneous equation (4). A second, linearly independent solution \( y_2(t) \) of eq. (4) can be found of the form \( y_2(t) = v(t)y_1(t) \), where the function \( v(t) \) is determined by replacing \( y_2 \) in eq. (4), leading to a first-order differential equation in \( v'(t) \).

Non-Homogeneous Equations

A general solution of the non-homogeneous equation (2) on an interval \( I \) can be written as

\[ y = y_p + y_{gen}^{hom} \]

where \( y_p \) is a particular solution of the non-homogeneous equation (2) and \( y_{gen}^{hom} \) is the general solution on \( I \) of the associated homogeneous equation (4).

\[ \rightarrow \] If the general solution \( y_{gen}^{hom} \) of the homogeneous equation (4) is known, a particular solution \( y_p \) of the non-homogeneous equation (2) can be found using the method of Variation of Parameters.
CAUCHY-EULER (EQUIDIMENSIONAL) EQUATIONS

A Cauchy-Euler equation is a second-order differential equation of the form:

\[ a t^2 y'' + b t y' + c y = f(t) \]  \hfill (5)

where \( a, b, c \) are real constants.

Homogeneous Cauchy-Euler Equations

A homogeneous Cauchy-Euler equation is of the form

\[ a t^2 y'' + b t y' + c y = 0 \]  \hfill (6)

where \( a, b, c \in \mathbb{R} \).

Looking for solutions of the form \( y = t^r \) of eq. (6) we obtain the associated characteristic equation:

\[ ar^2 + (b - a)r + c = 0 \]  \hfill (7)

Method for finding the general solution of the homogeneous equation (6):

- find the roots \( r_1, r_2 \) of the auxiliary equation (7);
- if \( r_1, r_2 \in \mathbb{R} \) and \( r_1 \neq r_2 \), then \( y(t) = c_1 t^{r_1} + c_2 t^{r_2} \) is the general solution of (6);
- if \( r_1 = r_2 = r \) then \( y(t) = t^r (c_1 + c_2 \ln t) \) is the general solution of (6);
- if \( r_1 = r_2 = \alpha + i \beta \in \mathbb{C} \setminus \mathbb{R} \) then \( y(t) = t^\alpha (c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)) \) is the solution of (6), where \( c_1 \) and \( c_2 \) are arbitrary real constants.